

REPRESENTATIONS OF LIE SUPERALGEBRAS IN PRIME CHARACTERISTIC III

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ABSTRACT. For a restricted Lie superalgebra \mathfrak{g} over an algebraically closed field of characteristic $p > 2$, we generalize the deformation method of Premet and Skryabin to obtain results on the p -power and 2-power divisibility of dimensions of \mathfrak{g} -modules. In particular, we give a new proof of the Super Kac-Weisfeiler conjecture for basic classical Lie superalgebras. The new proof allows us to improve optimally the assumption on p . We also establish a semisimplicity criterion for the reduced enveloping superalgebras associated with semisimple p -characters for all basic classical Lie superalgebras using the technique of odd reflections.

1. INTRODUCTION

1.1. In [WZ1], Wang and the author initiated the study of modular representation theory of Lie superalgebras over an algebraically closed field K of characteristic $p > 2$. Among other things, a superalgebra generalization (called Super KW conjecture) of celebrated Kac-Weisfeiler conjecture (Premet's Theorem) was formulated; and we established it for the most important class of Lie superalgebras—the basic classical Lie superalgebras, which are first classified over the complex numbers by Kac [Kac] and Scheunert-Nahm-Rittenberg [SNR]. Our work generalized the earlier work on Lie algebras of reductive algebraic groups by Kac-Weisfeiler [WK], Friedlander-Parshall [FP], Premet [Pr1, Pr2], Skryabin [Skr], and others (see Jantzen [Jan1] for an excellent review and extensive references on modular representations of Lie algebras).

In our proof of the super KW conjecture, a \mathbb{Z} -grading of the basic classical Lie superalgebras plays an important role. In order to obtain the grading, we imposed somewhat restrictive conditions on p [WZ1, Section 2.2].

In [PS], Premet and Skryabin developed deformation techniques by considering a family of \mathcal{L} -associative algebras for a restricted Lie algebra \mathcal{L} to derive results on dimensions of simple \mathcal{L} -modules. In particular, their method gives a new proof of the Kac-Weisfeiler conjecture which differs completely from Premet's original approach [Pr1].

1.2. The first main goal of this paper is to generalize some of the ideas in [PS] to the superalgebra setting. In particular, we provide a new proof of Super KW conjecture for basic classical Lie superalgebras so that the over-restrictive assumption on p in [WZ1, Section 2.2] is relaxed optimally.

Our second goal is to give a simplicity criterion for baby Verma modules as well as a semisimplicity criterion for reduced enveloping superalgebras of basic classical Lie superalgebras with semisimple p -characters.

1.3. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ be an $(n_0|n_1)$ -dimensional restricted Lie superalgebra over K and let $\xi \in \mathfrak{g}_{\bar{0}}^*$. Let $S(\mathfrak{g})$ be the symmetric superalgebra on \mathfrak{g} . The *reduced symmetric superalgebra* $S_{\xi}(\mathfrak{g})$ associated with ξ is defined to be the quotient of $S(\mathfrak{g})$ by the ideal generated by elements of the form $(x - \xi(x))^p$ with $x \in \mathfrak{g}_{\bar{0}}$. It is a local (super)commutative superalgebra of dimension $p^{n_0}2^{n_1}$. Let $U_{\xi}(\mathfrak{g})$ be the reduced enveloping superalgebra as usual.

Following [PS], we introduce a family of associative superalgebras $U_{\xi,\lambda}(\mathfrak{g})$, where $\xi \in \mathfrak{g}_{\bar{0}}^*$ and $\lambda \in K$, parametrized by the points of the projective space $\mathbb{P}(\mathfrak{g}_{\bar{0}}^* \oplus K)$ (the superalgebras $U_{t\xi,t\lambda}(\mathfrak{g})$ with $t \in K^{\times}$ being isomorphic). The Lie superalgebra \mathfrak{g} acts on each $U_{\xi,\lambda}(\mathfrak{g})$ as derivations. The family relates the reduced enveloping superalgebra $U_{\xi}(\mathfrak{g})$ ($= U_{\xi,1}(\mathfrak{g})$) to the reduced symmetric superalgebra $S_{\xi}(\mathfrak{g})$ ($= U_{\xi,0}(\mathfrak{g})$). As in the Lie algebra case [PS], the reduced symmetric superalgebra $S_{\xi}(\mathfrak{g})$ has favorable structures of \mathfrak{g} -invariant ideals (cf. Proposition 2.6).

Following [PS] but with slight modification, we use the method of associated cones in invariant theory to obtain some results on the $(p, 2)$ -divisibility of dimensions of \mathfrak{g} -modules. In particular, we show that (Theorem 3.2 (ii)) for an arbitrary restricted Lie superalgebra \mathfrak{g} and $\chi \in \mathfrak{g}_{\bar{0}}^*$, if

- (\star) all nonzero scalar multiples of χ are conjugate under the group $G(\mathfrak{g}_{\bar{0}})$ of automorphisms of $\mathfrak{g}_{\bar{0}}$ which preserve the restricted structure,

then the super KW conjecture holds for $U_{\chi}(\mathfrak{g})$. Note that (\star) is a non-super condition. Now if \mathfrak{g} is one of the basic classical Lie superalgebras as in Section 2.4 with the optimal assumption on p or the queer Lie superalgebra as in [WZ2] and if $\chi \in \mathfrak{g}_{\bar{0}}^*$ is nilpotent, then condition (\star) is satisfied ([Jan2, Sections 2.8, 2.10]). Thus the super KW conjecture for basic classical Lie superalgebras and the queer Lie superalgebra with nilpotent p -characters holds. Together with the Morita equivalence theorem [WZ1, Theorem 5.2], this gives a new proof of the super KW conjecture for basic classical Lie superalgebras in full generality with the optimal assumption on p .

1.4. For the reduced enveloping superalgebras of basic classical Lie superalgebras with semisimple p -characters, we give a simplicity criterion for baby Verma modules. As a consequence, we obtain a semisimplicity criterion for the reduced enveloping superalgebras. These results, first announced in [Z, Remark 4.5], generalize results of Rudakov [Rud] and Friedlander-Parshall [FP] for Lie algebras.

A major complication in the super case is due to the existence of non-conjugate sets of simple roots. We settle the problem by using the technique of odd reflections (see [Ser] for example). This approach is quite different from the proof of the corresponding results for type I basic classical Lie superalgebras in [Z].

In his paper [Zh], C. Zhang independently stated the simplicity criterion for baby Verma modules with semisimple p -characters for basic classical Lie superalgebras

(the statement of Theorem 4.6). However, his proof, which relied essentially on an erroneous lemma [Zh, Lemma 3.6], is incorrect.

1.5. The paper is laid out as follows. In Section 2, after reviewing some basic facts about modular representations of Lie superalgebras and basic classical Lie superalgebras, we introduce the super generalization of families of associative algebras following [PS]. Then we study the properties of invariant ideals of the reduced symmetric superalgebras. The new proof of super KW conjecture for basic classical Lie superalgebras is given in Section 3. Finally, Section 4 is devoted to the study of basic classical Lie superalgebras with semisimple p -characters.

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2. RESTRICTED LIE SUPERALGEBRAS AND FAMILIES OF \mathfrak{g} -SUPERALGEBRAS

2.1. Throughout we work with an algebraically closed field K with characteristic $p > 2$ as the ground field. We exclude $p = 2$ since in that case Lie superalgebras coincide with Lie algebras.

A superspace is a \mathbb{Z}_2 -graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, in which we call elements in $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) even (resp. odd). Write $|v| \in \mathbb{Z}_2$ for the parity (or degree) of $v \in V$, which is implicitly assumed to be (\mathbb{Z}_2) -homogeneous. A bilinear form f on V is *supersymmetric* if $f(u, v) = (-1)^{|u||v|} f(v, u)$ for all homogeneous $u, v \in V$. We will use the notation

$$\underline{\dim} V = \dim V_{\bar{0}} | \dim V_{\bar{1}}; \quad \dim V = \dim V_{\bar{0}} + \dim V_{\bar{1}}.$$

If W is a subsuperspace of V , denote

$$\underline{\text{codim}}_V W = \underline{\dim} V - \underline{\dim} W; \quad \text{codim}_V W = \dim V - \dim W.$$

Sometimes we simply write $\underline{\text{codim}} W$ and $\text{codim} W$ for short when the total space V is clear from the context.

All Lie superalgebras \mathfrak{g} will be assumed to be finite dimensional. We will use $U(\mathfrak{g})$ to denote its universal enveloping superalgebra.

According to Walls [W], the finite-dimensional simple associative superalgebras over K are classified into two types: besides the usual matrix superalgebra (called type M) there are in addition simple superalgebras of type Q .

By vector spaces, derivations, subalgebras, ideals, modules, submodules, and commutativity, etc. we mean in the super sense unless otherwise specified.

For a real number a , we use $\lfloor a \rfloor$ to denote its least integer upper bound, and use $\lceil a \rceil$ to denote its greatest integer lower bound.

2.2. Recall a restricted Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra whose even subalgebra $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra which admits a $[p]$ th power map $^{[p]} : \mathfrak{g}_{\bar{0}} \rightarrow \mathfrak{g}_{\bar{0}}$ satisfying certain conditions ([Jac, Chap. V]), and whose odd part $\mathfrak{g}_{\bar{1}}$ is a restricted module by the adjoint action of the even subalgebra $\mathfrak{g}_{\bar{0}}$.

All the Lie (super)algebras in this paper will be assumed to be restricted.

Let \mathfrak{g} be a restricted Lie superalgebra, for each $\chi \in \mathfrak{g}_0^*$, the *reduced enveloping superalgebra* of \mathfrak{g} with the p -character χ is by definition the quotient of $U(\mathfrak{g})$ by the ideal I_χ generated by all $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}_0$.

We further recall the definition of (super)derivations. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra. Then its endomorphism algebra $\text{End}_K(A)$ is naturally \mathbb{Z}_2 -graded with

$$\text{End}_K(A)_i = \{f \in \text{End}_K(A) \mid f(A_j) \subseteq A_{j+i}, \text{ for } j \in \mathbb{Z}_2\}, \quad i \in \mathbb{Z}_2.$$

Let $\text{Der}_i(A)$, $i \in \mathbb{Z}_2$, be the subspace of all $\delta \in \text{End}_K(A)_i$ such that

$$\delta(xy) = (\delta x)y + (-1)^{i|x|}x(\delta y)$$

for all homogeneous $x, y \in A$.

The Lie superalgebra of derivations of A

$$\text{Der}(A) = \text{Der}_{\bar{0}}(A) \oplus \text{Der}_{\bar{1}}(A)$$

is a restricted Lie subalgebra of $\text{End}_K(A)$.

2.3. Let \mathfrak{g} be a restricted Lie superalgebra. For $\chi \in \mathfrak{g}_0^*$, we always regard $\chi \in \mathfrak{g}^*$ by setting $\chi(\mathfrak{g}_{\bar{1}}) = 0$. Denote the centralizer of χ in \mathfrak{g} by $\mathfrak{g}_\chi = \mathfrak{g}_{\chi, \bar{0}} + \mathfrak{g}_{\chi, \bar{1}}$, where $\mathfrak{g}_{\chi, i} = \{y \in \mathfrak{g}_i \mid \chi([y, \mathfrak{g}]) = 0\}$ for $i \in \mathbb{Z}_2$. Set $d_0 \mid d_1 = \underline{\text{codim}} \mathfrak{g}_\chi$. It is well-known that d_0 is even whereas d_1 could be odd.

We recall here the following superalgebra generalization of the Kac-Weisfeiler Conjecture, which is formulated in [WZ1].

Super KW Conjecture. *The dimension of every $U_\chi(\mathfrak{g})$ -module is divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$.*

2.4. The basic classical Lie superalgebras over the complex field \mathbb{C} were classified independently by Kac [Kac], and Scheunert-Nahm-Rittenberg [SNR]. Those Lie superalgebras by definition admit an even nondegenerate supersymmetric bilinear form, and the even subalgebras are reductive.

We observe that the basic classical Lie superalgebras are defined over fields of positive characteristics as well under mild assumption on p (see [WZ1, Sect. 2]). The restriction on the characteristic of fields of definition is listed in the following table (the general linear Lie superalgebra, though not simple, is also included).

Lie superalgebra	Characteristic of K
$\mathfrak{gl}(m n)$	$p > 2$
$\mathfrak{sl}(m n)$	$p > 2, p \nmid (m - n)$
$B(m, n), C(n), D(m, n)$	$p > 2$
$D(2, 1; \alpha)$	$p > 3$
$F(4)$	$p > 2$
$G(3)$	$p > 3$

TABLE: basic classical Lie K -superalgebras

Note that for each basic classical Lie superalgebra \mathfrak{g} , the restriction on the prime p above makes p automatically *good* for the even subalgebra \mathfrak{g}_0 (cf. [Jan2, Section 2.6]).

2.5. In the following two subsections, we introduce, following [PS], a family of associative superalgebras deformed from the reduced enveloping superalgebras. This part can be viewed as a super counterpart of [PS, Sect. 2]; since the proofs of the statements are essentially the same as those of the corresponding ones in *loc. cit.*, we will omit them and refer the reader to the original paper.

Let \mathfrak{g} be a $n_0|n_1$ -dimensional restricted Lie superalgebra. A \mathfrak{g} -superalgebra is a pair consisting of a K -superalgebra A and a homomorphism $\mathfrak{g} \rightarrow \text{Der } A$ of restricted Lie superalgebras.

Given a linear form $\xi \in \mathfrak{g}_0^*$ and a scalar $\lambda \in K$, denote by $U_{\xi, \lambda}(\mathfrak{g})$ the quotient superalgebra of the tensor superalgebra $T(\mathfrak{g})$ on the superspace \mathfrak{g} by its ideal $I_{\xi, \lambda}$ generated by all elements $x \otimes y - (-1)^{|x||y|}y \otimes x - \lambda[x, y]$ for all homogeneous $x, y \in \mathfrak{g}$ and elements $x^{\otimes p} - \lambda^{p-1}x^{[p]} - \xi(x)^p \cdot 1$ for all $x \in \mathfrak{g}_0$. Each $U_{\xi, \lambda}(\mathfrak{g})$ is a \mathfrak{g} -superalgebra.

If $\lambda = 1$, the superalgebra $U_{\xi, \lambda}(\mathfrak{g})$ is the reduced enveloping superalgebra $U_{\xi}(\mathfrak{g})$; while if $\lambda = 0$, the superalgebra is called the *reduced symmetric superalgebra*, denoted by $S_{\xi}(\mathfrak{g})$. Since $x^p - \xi(x)^p = (x - \xi(x))^p$ for $x \in \mathfrak{g}_0$, by changing of variables we see that $S_{\xi}(\mathfrak{g})$ is isomorphic to the truncated polynomial superalgebra

$$K[x_1, \dots, x_{n_0}; y_1, \dots, y_{n_1}] / (x_1^p, \dots, x_{n_0}^p; y_1^2, \dots, y_{n_1}^2),$$

where $K[x_1, \dots, x_{n_0}; y_1, \dots, y_{n_1}]$ is the (free) commutative superalgebra on even generators $\{x_1, \dots, x_{n_0}\}$ and odd generators $\{y_1, \dots, y_{n_1}\}$. The unique maximal ideal of $S_{\xi}(\mathfrak{g})$ is generated by all $x - \xi(x) \cdot 1$ for $x \in \mathfrak{g}_0$ and all $y \in \mathfrak{g}_1$.

If $t \in K^{\times} = K \setminus \{0\}$, the map $x \mapsto t^{-1}x$, where $x \in \mathfrak{g}$, extends uniquely to the superalgebra isomorphism

$$\theta_t : U_{\xi, \lambda}(\mathfrak{g}) \rightarrow U_{t\xi, t\lambda}(\mathfrak{g}).$$

In particular, if $\lambda \neq 0$, then $U_{\xi, \lambda}(\mathfrak{g}) \cong U_{\lambda^{-1}\xi}(\mathfrak{g})$ as superalgebras. All superalgebra isomorphism θ_t are \mathfrak{g} -equivariant.

2.6. A vector bundle $A \rightarrow Z$ over an algebraic variety Z together with a pair of morphism $\mu : A \times_Z A \rightarrow A$ and $\rho : \mathfrak{g} \times A \rightarrow A$ of algebraic varieties over Z is called a *continuous family of (finite-dimensional) \mathfrak{g} -superalgebras parametrized by Z* if, for the fiber A_{ζ} over any point $\zeta \in Z$,

- (1) the restriction of μ to $A_{\zeta} \times A_{\zeta}$ gives A_{ζ} a structure of a finite-dimensional associative superalgebra.
- (2) the restriction of ρ to $\mathfrak{g} \times A_{\zeta}$ induces a homomorphism of restricted Lie superalgebras $\mathfrak{g} \rightarrow \text{Der } A_{\zeta}$.

The algebraic variety Z is called the parameter space of the family. By definition, all \mathfrak{g} -superalgebras in a family have the same finite dimension.

The isomorphisms θ_t allow us pass to a continuous family of superalgebras parametrized by the projective space $\mathbb{P}(\mathfrak{g}_0^* \oplus K)$ corresponding to the linear space

$\mathfrak{g}_0^* \oplus K$. Write $(\xi : \lambda)$ for the point of $\mathbb{P}(\mathfrak{g}_0^* \oplus K)$ represented by the pair $(\xi, \lambda) \neq (0, 0)$, where $\xi \in \mathfrak{g}_0^*$, $\lambda \in K$. Identify $\mathbb{P}(\mathfrak{g}_0^*)$ with the Zariski closed subset of $\mathbb{P}(\mathfrak{g}_0^* \oplus K)$ consisting of all points $(\xi : \lambda)$ with $\lambda = 0$. Identify each $\xi \in \mathfrak{g}_0^*$ with the point $(\xi : 1) \in \mathbb{P}(\mathfrak{g}_0^* \oplus K)$.

Proposition 2.1. *The set of superalgebras $U_{\xi, \lambda}(\mathfrak{g})$ with $(\xi : \lambda) \in \mathbb{P}(\mathfrak{g}_0^* \oplus K)$ is a continuous family of \mathfrak{g} -superalgebras parametrized by $\mathbb{P}(\mathfrak{g}_0^* \oplus K)$ such that the superalgebras corresponding to the points $\xi \in \mathfrak{g}_0^*$ and $(\xi : 0) \in \mathbb{P}(\mathfrak{g}_0^*)$ of the parameter space are \mathfrak{g} -equivariantly isomorphic to $U_{\xi}(\mathfrak{g})$ and $S_{\xi}(\mathfrak{g})$, respectively.*

Proof. The proof is the same as that of [PS, Proposition 2.2], and will be omitted here. \square

Lemma 2.2. *Let $\pi : A \rightarrow Z$ be a continuous family of \mathfrak{g} -superalgebras parametrized by an algebraic variety Z . Then, for any positive integer d , the set of all points $\zeta \in Z$ such that the corresponding superalgebra A_{ζ} contains a \mathfrak{g} -invariant two-sided ideal of dimension d is closed in Z .*

Proof. For a superspace V , let $\sigma : V \rightarrow V$ be the linear transformation whose action on the homogeneous elements is given by

$$\sigma(v) = (-1)^{|v|}v.$$

Then a subspace W of V is graded if and only if $\sigma(W) = (W)$.

Let $\varphi : G_d(A) \rightarrow Z$ be the Grassmann bundle of d -dimensional subspaces corresponding to the vector bundle $\pi : A \rightarrow Z$. Then the subvariety $G_d^{\text{gr}}(A) \subseteq G_d(A)$ of graded subspaces of dimension d is closed.

Given this, the rest of the proof is the same as the proof of [PS, Lemma 2.3]. \square

2.7. In the rest of this section, we study the properties of invariant ideals of the reduced symmetric superalgebras. This can be viewed as the super counterpart of [PS, Section 3]. It turns out that most statements and their proofs in *loc. cit.* generalize to the super setup trivially. As we did in the previous two subsections, we will only state the facts without proof when their proofs are straightforward generalization of the corresponding ones in *loc. cit.*

Let \mathfrak{p} be a restricted subalgebra of \mathfrak{g} , and $\xi \in \mathfrak{g}_0^*$. For any $U_{\xi}(\mathfrak{p})$ -module V , the superspace

$$\tilde{V} = \text{Hom}_{U_{\xi}(\mathfrak{p})}(U_{\xi}(\mathfrak{g}), V)$$

carries a standard $U_{\xi}(\mathfrak{g})$ -module structure given by

$$(xf)(v) = (-1)^{|x|(|f|+|v|)}f(vx)$$

where $x, v \in U_{\xi}(\mathfrak{g})$, and $f \in \text{Hom}_{U_{\xi}(\mathfrak{p})}(U_{\xi}(\mathfrak{g}), V)$ are homogeneous elements. This module is called the $U_{\xi}(\mathfrak{g})$ -module *coinduced* from V .

Let A be a \mathfrak{p} -superalgebra. The restricted \mathfrak{g} -module $\tilde{A} = \text{Hom}_{U_0(\mathfrak{p})}(U_0(\mathfrak{g}), A)$ coinduced from A carries a superalgebra structure such that the \mathfrak{g} acts on \tilde{A} as superderivations. The multiplication in \tilde{A} is given by the formula

$$(f \cdot g)(u) = \sum_{(u)} (-1)^{|g||u_{(1)}|} f(u_{(1)})g(u_{(2)}),$$

where $f, g \in \tilde{A}$ and $u \in U_0(\mathfrak{g})$ are homogenous, and where $u \mapsto \sum u_{(1)} \otimes u_{(2)}$ is the comultiplication of $U_0(\mathfrak{g})$.

2.8. Let \mathfrak{p} be a restricted subalgebra of \mathfrak{g} . Write

$$\mathcal{F}(\mathfrak{g}, \mathfrak{p}) = \text{Hom}_{U_0(\mathfrak{p})}(U_0(\mathfrak{g}), K),$$

where K denotes the trivial $U_0(\mathfrak{p})$ -module.

Lemma 2.3. *The superalgebra $\mathcal{F}(\mathfrak{g}, \mathfrak{p})$ is \mathfrak{g} -simple and commutative. Moreover, it is isomorphic to a truncated symmetric superalgebra. The unique maximal ideal $\mathfrak{m}(\mathfrak{g}, \mathfrak{p})$ of $\mathcal{F}(\mathfrak{g}, \mathfrak{p})$ consists of all $f \in \mathcal{F}(\mathfrak{g}, \mathfrak{p})$ satisfying $f(1) = 0$.*

Proof. Let $\{x_1, \dots, x_s\}$ (resp. $\{y_1, \dots, y_t\}$) be elements in \mathfrak{g}_0 (resp. \mathfrak{g}_1) such that their images form a basis for $\mathfrak{g}_0/\mathfrak{p}_0$ (resp. $\mathfrak{g}_1/\mathfrak{p}_1$).

Let

$$\Lambda_{\bar{0}} = \{\mathbf{a} = (a_1, \dots, a_s) \mid 0 \leq a_i \leq p-1 \text{ are integers}\};$$

$$\Lambda_{\bar{1}} = \{\mathbf{b} = (b_1, \dots, b_r) \mid 1 \leq b_1 < \dots < b_r \leq t \text{ are integers}; 0 \leq r \leq t\}.$$

For $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{a}' = (a'_1, \dots, a'_s)$ in $\Lambda_{\bar{0}}$, denote $\mathbf{a}! = \prod (a_i!)$. Write $\mathbf{a}' \leq \mathbf{a}$ if $a'_i \leq a_i$ for all i . Further put $\binom{\mathbf{a}}{\mathbf{a}'} = \prod \binom{a_i}{a'_i}$ when $\mathbf{a}' \leq \mathbf{a}$. For $\mathbf{b} = (b_1, \dots, b_r)$ and $\mathbf{b}' = (b'_1, \dots, b'_l)$ in $\Lambda_{\bar{1}}$, write $\mathbf{b}' \leq \mathbf{b}$ if (b'_1, \dots, b'_l) appears in (b_1, \dots, b_r) as a subsequence. Also, when $\mathbf{b}' \leq \mathbf{b}$, define $\text{sgn}(\mathbf{b}', \mathbf{b})$ to be the sign of the permutation of sequence \mathbf{b} given by $(\mathbf{b}', \mathbf{b} \setminus \mathbf{b}')$, where $\mathbf{b} \setminus \mathbf{b}'$ denotes the subsequence of \mathbf{b} formed by removing the subsequence \mathbf{b}' from \mathbf{b} .

For $\mathbf{a} = (a_1, \dots, a_s) \in \Lambda_{\bar{0}}$ and $\mathbf{b} = (b_1, \dots, b_r) \in \Lambda_{\bar{1}}$, write

$$e^{(\mathbf{a}, \mathbf{b})} = x_1^{a_1} \cdots x_s^{a_s} y_{b_1} \cdots y_{b_r}.$$

Then $U_0(\mathfrak{g})$ is a free $U_0(\mathfrak{p})$ -module on basis

$$\{e^{(\mathbf{a}, \mathbf{b})} \mid \mathbf{a} \in \Lambda_{\bar{0}}, \mathbf{b} \in \Lambda_{\bar{1}}\}.$$

The comultiplication of $U_0(\mathfrak{g})$ on $e^{(\mathbf{a}, \mathbf{b})}$ is given by

$$\Delta(e^{(\mathbf{a}, \mathbf{b})}) = \sum_{\mathbf{a}' \leq \mathbf{a}; \mathbf{b}' \leq \mathbf{b}} \binom{\mathbf{a}}{\mathbf{a}'} \text{sgn}(\mathbf{b}', \mathbf{b}) e^{(\mathbf{a}', \mathbf{b}')} \otimes e^{(\mathbf{a} - \mathbf{a}', \mathbf{b} \setminus \mathbf{b}')}. \quad (2.1)$$

Let $\phi_i \in \mathcal{F}(\mathfrak{g}, \mathfrak{p})_{\bar{0}}$ (resp. $\psi_j \in \mathcal{F}(\mathfrak{g}, \mathfrak{p})_{\bar{1}}$) be the dual element of x_i for $1 \leq i \leq s$ (resp. y_j for $1 \leq j \leq t$).

Equation (2.1) inductively shows that, for $\mathbf{a}, \mathbf{a}' \in \Lambda_{\bar{0}}$ and $\mathbf{b}, \mathbf{b}' \in \Lambda_{\bar{1}}$,

$$\phi^{\mathbf{a}} \psi^{\mathbf{b}}(e^{(\mathbf{a}', \mathbf{b}')}) = \mathbf{a}! \delta_{(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')},$$

where we use the notation

$$\phi^{\mathbf{a}} = \phi_1^{a_1} \cdots \phi_s^{a_s}, \quad \psi^{\mathbf{b}} = \psi_{b_1} \cdots \psi_{b_r},$$

for $\mathbf{a} = (a_1, \dots, a_s)$, $\mathbf{b} = (b_1, \dots, b_r)$.

Then $\mathcal{F}(\mathfrak{g}, \mathfrak{p})$ is an associative superalgebra with unit element and generators ϕ_1, \dots, ϕ_s and ψ_1, \dots, ψ_t , which satisfy $\phi_i^p = 0$ and $\psi_j^2 = 0$ for all i, j . As its dimension is $p^s 2^t$, there is an isomorphism

$$K[x_1, \dots, x_s; y_1, \dots, y_t]/(x_1^p, \dots, x_s^p; y_1^2, \dots, y_t^2) \cong \mathcal{F}(\mathfrak{g}, \mathfrak{p}).$$

To see it is \mathfrak{g} -simple, we note inductively from equation (2.1) that the action of \mathfrak{g} on some of the basis vectors of $\mathcal{F}(\mathfrak{g}, \mathfrak{p})$ is given as follows:

$$\begin{aligned} & x_i \cdot \phi_1^{a_1} \cdots \phi_s^{a_s} \psi^{\mathbf{b}} \\ &= \begin{cases} 0 & \text{if } a_i = 0, \\ \lambda \phi_1^{a_1} \cdots \phi_i^{a_i-1} \cdots \phi_s^{a_s} \psi^{\mathbf{b}} & \text{if } 2 \leq a_i \leq p-1, \\ \mu \phi_1^{a_1} \cdots \phi_{i-1}^{a_{i-1}} \phi_{i+1}^{a_{i+1}} \cdots \phi_s^{a_s} \psi^{\mathbf{b}} + \nu \phi_1^{a_1} \cdots \phi_i^{p-1} \cdots \phi_s^{a_s} \psi^{\mathbf{b}} & \text{if } a_i = 1; \end{cases} \\ & y_j \cdot \psi_{j_1} \cdots \psi_{j_r} \\ &= \begin{cases} 0 & \text{if } j \notin \{j_1, \dots, j_r\}, \\ \pm \psi_{j_1} \cdots \hat{\psi}_j \cdots \psi_{j_r} & \text{otherwise.} \end{cases} \end{aligned}$$

where λ, μ , and ν are in K with λ, μ nonzero.

Given any nonzero element in $\mathcal{F}(\mathfrak{g}, \mathfrak{p})$, by applying a suitable sequence of x_i 's and y_j 's, we will eventually arrive at a linear combination of basis vectors $\phi^{\mathbf{a}} \psi^{\mathbf{b}}$ with nonzero constant term. On the other hand, since $\phi_i^p = 0$ and $\psi_j^2 = 0$, every nonzero \mathfrak{g} -invariant ideal is nilpotent and contains an element with nonzero constant term. It has to be the whole thing. Hence $\mathcal{F}(\mathfrak{g}, \mathfrak{p})$ is \mathfrak{g} -simple. The rest of the statement is clear. \square

Let $B = B_{\bar{0}} \oplus B_{\bar{1}}$ be a finite dimensional unital commutative associative \mathfrak{g} -superalgebra. The superalgebra B is said to be \mathfrak{g} -simple if it contains no nonzero \mathfrak{g} -invariant ideals. Arguing as in [PS, 3.2], we can show that if B is \mathfrak{g} -simple, then it is a local superalgebra with the unique maximal ideal $\mathfrak{m} = \mathfrak{m}_{\bar{0}} \oplus B_{\bar{1}}$, where $\mathfrak{m}_{\bar{0}}$ consists of the elements $b \in B_{\bar{0}}$ such that $b^p = 0$.

Proposition 2.4. *Let B be a \mathfrak{g} -simple finite dimensional unital commutative \mathfrak{g} -superalgebra. Denote by \mathfrak{m} the maximal ideal, and by \mathfrak{p} the normalizer of \mathfrak{m} in \mathfrak{g} . Then there is a canonical \mathfrak{g} -equivariant superalgebra isomorphism $B \cong \mathcal{F}(\mathfrak{g}, \mathfrak{p})$.*

Proof. The proof is similar to the proof of [PS, Thm. 3.2], and will be skipped here. \square

2.9. Let B be a commutative \mathfrak{g} -superalgebra and $\xi \in \mathfrak{g}_0^*$. By a $(B, U_\xi(\mathfrak{g}))$ -module, we mean a $U_\xi(\mathfrak{g})$ -module which is also a module over superalgebra B such that the module structure map $B \otimes M \rightarrow M$ is a \mathfrak{g} -module homomorphism. A (B, \mathfrak{g}) -superalgebra is a K -superalgebra C , which is simultaneously a B -superalgebra and \mathfrak{g} -superalgebra and a $(B, U_0(\mathfrak{g}))$ -module.

Now let $B = \mathcal{F}(\mathfrak{g}, \mathfrak{p})$. For any $U_\xi(\mathfrak{p})$ -module V , the coinduced $U_\xi(\mathfrak{g})$ -module \tilde{V} carries a canonical $(\mathcal{F}(\mathfrak{g}, \mathfrak{p}), U_\xi(\mathfrak{g}))$ -module structure given by

$$(f \cdot \psi)(u) = \sum_{(u)} (-1)^{|\psi||u_{(1)}|} f(u_{(1)}) \psi(u_{(2)}),$$

where $f \in \mathcal{F}(\mathfrak{g}, \mathfrak{p})$, $\psi \in \tilde{V}$, and $u \in U_\xi(\mathfrak{g})$ are homogeneous.

If A is a \mathfrak{p} -superalgebra, then the $(\mathcal{F}(\mathfrak{g}, \mathfrak{p}), U_0(\mathfrak{g}))$ -module $\tilde{A} = \text{Hom}_{U_0(\mathfrak{p})}(U_0(\mathfrak{g}), A)$ has a \mathfrak{g} -invariant multiplication, it is $\mathcal{F}(\mathfrak{g}, \mathfrak{p})$ -bilinear as well. Therefore, \tilde{A} is an $(\mathcal{F}(\mathfrak{g}, \mathfrak{p}), \mathfrak{g})$ -superalgebra.

Proposition 2.5. *Let M be an $(\mathcal{F}(\mathfrak{g}, \mathfrak{p}), U_\xi(\mathfrak{g}))$ -module and C an $(\mathcal{F}(\mathfrak{g}, \mathfrak{p}), \mathfrak{g})$ -superalgebra. Then,*

- (i) $M \cong \text{Hom}_{U_\xi(\mathfrak{p})}(U_\xi(\mathfrak{g}), M/\mathfrak{m}(\mathfrak{g}, \mathfrak{p})M)$ as $(\mathcal{F}(\mathfrak{g}, \mathfrak{p}), U_\xi(\mathfrak{g}))$ -module.
- (ii) $C \cong \text{Hom}_{U_0(\mathfrak{p})}(U_0(\mathfrak{g}), C/\mathfrak{m}(\mathfrak{g}, \mathfrak{p})C)$ as $(\mathcal{F}(\mathfrak{g}, \mathfrak{p}), \mathfrak{g})$ -superalgebras.

Proof. The proof is similar to proof of [PS, Thm 3.3], and will be skipped here. \square

2.10. Let $\xi \in \mathfrak{g}_0^*$. Recall the centralizer of ξ in \mathfrak{g} is denoted by \mathfrak{g}_ξ , which is a restricted Lie subalgebra. Put $d_0 | d_1 = \underline{\text{codim}} \mathfrak{g}_\xi$.

Proposition 2.6. *Let $\xi \in \mathfrak{g}_0^*$ and $d_0 | d_1 = \underline{\text{codim}} \mathfrak{g}_\xi$. Then each \mathfrak{g} -invariant ideal of $S_\xi(\mathfrak{g})$ has codimension divisible by $p^{d_0} 2^{d_1}$. Among them, there is a unique maximal one of codimension $p^{d_0} 2^{d_1}$.*

Proof. The proof, which uses Propositions 2.4 and 2.5(i), is similar to proof of [PS, Thm 3.4], and will be skipped here. \square

Remark 2.7. Let \mathcal{L} be an n -dimensional restricted Lie algebra, and let r be the minimal dimension of the centralizers of all $\chi \in \mathcal{L}^*$. It is conjectured by Kac-Weisfeiler that the maximal dimension $M(\mathcal{L})$ of simple \mathcal{L} -modules is $p^{\frac{n-r}{2}}$. Following [PS], we refer to this conjecture as KW1 conjecture, which is still open.

In [PS], Premet and Skryabin showed that

- (†) the set of $\chi \in \mathcal{L}^*$ such that $U_\chi(\mathcal{L})$ has all its simple modules having the maximal dimension $M(\mathcal{L})$ is nonempty and Zariski open in \mathcal{L}^* .

Using deformation arguments, they then showed that

- (‡) if there is $\chi \in \mathcal{L}^*$ whose centralizer is a toral subalgebra of \mathcal{L} , then there is a nonempty and Zariski open subset W of \mathcal{L}^* such that $\xi \in W$ implies that all simple $U_\xi(\mathcal{L})$ -modules have dimension $p^{\frac{n-r}{2}}$.

This, together with (†), confirms KW1 conjecture for such \mathcal{L} .

Along the line in this section, we can establish the corresponding statement of (‡) in the superalgebra setting. However, it is not clear how to generalize (†) to a general restricted Lie superalgebra. This is mainly due to the fact that the universal enveloping superalgebra is in general not a prime ring (see [B] for a counterexample over the complex numbers), which is crucial in the proof of (†) in [PS].

3. PROOF OF SUPER KW PROPERTY FOR BASIC CLASSICAL LIE SUPERALGEBRAS

3.1. In this subsection we first recall some basic facts on the method of associated cones, following [PS, Sect. 5.1].

Let V be a finite dimensional vector space over K . For an ideal I of the symmetric algebra $S(V^*)$, let $\text{gr}I$ denote the homogeneous ideal of $S(V^*)$ with the property that $g \in \text{gr}I \cap S^r(V^*)$ if and only if there is $\tilde{g} \in I$ such that

$$\tilde{g} - g \in \oplus_{j < r} S^j(V^*).$$

Identify $S(V^*)$ with the algebra of polynomial functions on V . Given a subset $X \subseteq V$, let

$$I_X = \{g \in S(V^*) \mid g(X) = 0\}$$

be the ideal associated to it. The set

$$\mathbb{K}X := \{v \in V \mid f(v) = 0 \text{ for all } f \in \text{gr}I_X\}.$$

is called the *cone associated with X* . It is a Zariski closed conical subset of V . We identify V (resp. $\mathbb{P}(V)$) with the subset of $\mathbb{P}(V \oplus K)$ consisting of all points $(v : 1)$ (resp., $(v : 0)$) with $v \in V$ (resp. $v \in V \setminus \{0\}$). Let \overline{X}^P (resp. \overline{X}) denote the Zariski closure of X in $\mathbb{P}(V \oplus K)$ (resp., in V). The following facts are easy to prove

$$\overline{X}^P \cap V = \overline{X} \text{ and } \overline{X}^P \cap \mathbb{P}(V) = \mathbb{P}(\mathbb{K}X), \quad (3.1)$$

where $\mathbb{P}(\mathbb{K}X) \subseteq \mathbb{P}(V)$ denotes the projectivisation of the conical subset $\mathbb{K}X$.

3.2. Now let \mathfrak{g} be a restricted Lie superalgebra. For a pair of nonnegative integers $(d_0 \mid d_1)$ with d_0 even, let \mathcal{X}_{d_0, d_1} denote the set of all $\xi \in \mathfrak{g}_0^*$ such that the algebra $U_\xi(\mathfrak{g})$ has a module of finite dimension not divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$, and let $\mathcal{X}'_{d_0, d_1} \subseteq \mathbb{P}(\mathfrak{g}_0^* \oplus K)$ be the subset of all points $(\xi : \lambda)$ satisfying $U_{\xi, \lambda}(\mathfrak{g})$ has a \mathfrak{g} -invariant ideal of codimension not divisible by $p^{d_0} 2^{d_1}$. Set

$$\mathcal{Y}_{d_0, d_1} = \{\xi \in \mathfrak{g}_0^* \mid \text{codim}_{\mathfrak{g}_{\bar{0}}} \mathfrak{g}_{\xi, \bar{0}} < d_0 \text{ or } \text{codim}_{\mathfrak{g}_{\bar{1}}} \mathfrak{g}_{\xi, \bar{1}} < d_1\}.$$

Note $\mathcal{X}_{d_0, 2k+1} = \mathcal{X}_{d_0, 2k+2}$, but this is not the case for \mathcal{X}'_{d_0, d_1} and \mathcal{Y}_{d_0, d_1} .

By Lemma 2.2, \mathcal{X}'_{d_0, d_1} is closed. The set \mathcal{Y}_{d_0, d_1} is obviously conical, and let $\mathbb{P}(\mathcal{Y}_{d_0, d_1}) \subseteq \mathbb{P}(\mathfrak{g}_0^*)$ be its projectivization. By Proposition 2.6, $\eta \in \mathfrak{g}_0^*$ lies in \mathcal{Y}_{d_0, d_1} if and only if $S_\eta(\mathfrak{g})$ has a \mathfrak{g} -invariant ideal with codimension not divisible by $p^{d_0} 2^{d_1}$. Therefore,

$$\mathcal{X}'_{d_0, d_1} \cap \mathbb{P}(\mathfrak{g}_0^*) = \mathbb{P}(\mathcal{Y}_{d_0, d_1}). \quad (3.2)$$

Hence $\mathbb{P}(\mathcal{Y}_{d_0, d_1})$ is closed in $\mathbb{P}(\mathfrak{g}_0^*)$, and so \mathcal{Y}_{d_0, d_1} is Zariski closed in \mathfrak{g}_0^* .

Proposition 3.1. *We have $\mathbb{K}\mathcal{X}_{d_0, d_1} \subseteq \mathcal{Y}_{d_0, d_1}$ for any pair of nonnegative integers $(d_0 \mid d_1)$ with d_0 even.*

Proof. We claim that $\mathcal{X}_{d_0, d_1} \subseteq \mathcal{X}'_{d_0, d_1} \cap \mathfrak{g}_0^*$. Indeed, suppose $\xi \in \mathfrak{g}_0^* \setminus (\mathcal{X}'_{d_0, d_1} \cap \mathfrak{g}_0^*)$. Then each two-sided ideal of $U_\xi(\mathfrak{g})$ is of codimension divisible by $p^{d_0} 2^{d_1}$ since all the two-sided ideals of $U_\xi(\mathfrak{g})$ are \mathfrak{g} -invariant. Let V be a simple module of $U_\xi(\mathfrak{g})$ and let $J = \text{Ann}_{U_\xi(\mathfrak{g})} V$ be its annihilator in $U_\xi(\mathfrak{g})$. Then by [LBF, Section 2], $U_\xi(\mathfrak{g})/J$ is a simple superalgebra over K with the unique simple module V since K is algebraically closed. Then either (1) $U_\xi(\mathfrak{g})/J$ is of type M with dimension a^2 for some natural number a ; or it is of type Q with dimension $2b^2$ for some natural number b . In case (1), the dimension of V is a . Then since $p^{d_0} 2^{d_1}$ divides $a^2 =$

$\dim U_\xi(\mathfrak{g})/J$, we will have $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$ divides a . In case (2), the dimension of V is $2b$. Since $p^{d_0} 2^{d_1}$ divides $2b^2 = \dim U_\xi(\mathfrak{g})/J$, $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$ divides b . In either case, the $U_\xi(\mathfrak{g})$ -module V has dimension divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$, which implies $\xi \notin \mathcal{X}_{d_0, d_1}$. The claim is proved.

The claim implies that $\overline{\mathcal{X}}_{d_0, d_1}^P \subseteq \mathcal{X}'_{d_0, d_1}$, since \mathcal{X}'_{d_0, d_1} is closed by Lemma 2.2. Then we have $\overline{\mathcal{X}}_{d_0, d_1}^P \cap \mathbb{P}(\mathfrak{g}_0^*) \subseteq \mathcal{X}'_{d_0, d_1} \cap \mathbb{P}(\mathfrak{g}_0^*)$, this means $\mathbb{P}(\mathbb{K}\mathcal{X}_{d_0, d_1}) \subseteq \mathbb{P}(\mathcal{Y}_{d_0, d_1})$ by (3.1) and (3.2). But since both $\mathbb{K}\mathcal{X}_{d_0, d_1}$ and \mathcal{Y}_{d_0, d_1} are conical, we deduce that $\mathbb{K}\mathcal{X}_{d_0, d_1} \subseteq \mathcal{Y}_{d_0, d_1}$, as desired. \square

3.3. Let $G(\mathfrak{g}_0)$ denote the group of all automorphisms of \mathfrak{g}_0 preserving the $[p]$ th power map, i.e., automorphisms g satisfying $g(x^{[p]}) = g(x)^{[p]}$ for all $x \in \mathfrak{g}_0$. Let $\Omega(\eta)$ denote the $G(\mathfrak{g}_0)$ -orbit of $\eta \in \mathfrak{g}_0^*$.

For $\chi \in \mathfrak{g}_0^*$, define

$$l_0(\chi) = \min_{\xi \in \mathbb{K}\Omega(\chi)} \dim \mathfrak{g}_{\xi, \bar{0}},$$

$$l_1(\chi) = \min_{\xi \in \mathbb{K}\Omega(\chi)} \dim \mathfrak{g}_{\xi, \bar{1}}.$$

Theorem 3.2. *Let \mathfrak{g} be an $(n_0 | n_1)$ -dimensional restricted Lie superalgebra, and $\chi \in \mathfrak{g}_0^*$. Write $l_i = l_i(\chi)$ for $i \in \mathbb{Z}_2$, and $d_0 | d_1 = \text{codim}_{\mathfrak{g}} \mathfrak{g}_\chi$. Then,*

- (i) *Each finite dimensional $U_\chi(\mathfrak{g})$ -module has dimension divisible by $p^{\frac{n_0 - l_0}{2}} 2^{\lfloor \frac{n_1 - l_1}{2} \rfloor}$.*
- (ii) *If all nonzero scalar multiples of χ are $G(\mathfrak{g}_0)$ -conjugate, then the dimensions of all finite dimensional $U_\chi(\mathfrak{g})$ -modules are divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$, i.e. the super KW conjecture holds for the algebra $U_\chi(\mathfrak{g})$.*

Proof. To prove part (i), we treat the p - and 2-divisibility separately. Suppose that $U_\xi(\mathfrak{g})$ has a finite dimensional module V such that $\dim V$ is not divisible by $2^{\lfloor \frac{n_1 - l_1}{2} \rfloor}$. Then $\chi \in \mathcal{X}_{0, n_1 - l_1}$ by the definition of $\mathcal{X}_{0, n_1 - l_1}$. (Note that in addition, $\chi \in \mathcal{X}_{0, n_1 - l_1 + 1}$ when $n_1 - l_1$ is odd, while $\chi \in \mathcal{X}_{0, n_1 - l_1 - 1}$ when $n_1 - l_1$ is even. But we do not need this.) Since for any $g \in G(\mathfrak{g}_0)$, the algebras $U_{g(\chi)}(\mathfrak{g})$ and $U_\chi(\mathfrak{g})$ are isomorphic. It follows that $\Omega(\chi) \subseteq \mathcal{X}_{0, n_1 - l_1}$. But then $\mathbb{K}\Omega(\chi) \subseteq \mathbb{K}\mathcal{X}_{0, n_1 - l_1}$. As $\mathbb{K}\mathcal{X}_{0, n_1 - l_1} \subseteq \mathcal{Y}_{0, n_1 - l_1}$ by Proposition 3.1, we have

$$\text{codim}_{\mathfrak{g}_1} \mathfrak{g}_{\xi, \bar{1}} < n_1 - l_1$$

for any $\xi \in \mathbb{K}\Omega(\chi)$, which contradicts the choice of l_1 .

The p -divisibility can be proved similarly.

For part (ii), note first that $(\chi : 0) \in \overline{K^\times \chi}^P$. Since by assumption that $K^\times \chi$ is contained in a single $G(\mathfrak{g}_0^*)$ -orbit, we have, by equation (3.1),

$$(\chi : 0) \in \overline{K^\times \chi}^P \cap \mathbb{P}(\mathfrak{g}_0^*) \subseteq \overline{\Omega(\chi)}^P \cap \mathbb{P}(\mathfrak{g}_0^*) = \mathbb{P}(\mathbb{K}\Omega(\chi)).$$

Thus $\chi \in \mathbb{K}\Omega(\chi)$, and as a result, $l_i \leq n_i - d_i$ for $i \in \mathbb{Z}_2$. From here, (ii) follows from (i). \square

3.4. Now let \mathfrak{g} be one of the basic classical Lie superalgebras as in Section 2.4. Recall that the even subalgebra \mathfrak{g}_0 is the Lie algebra of a reductive group G_0 , and that \mathfrak{g} admits an even nondegenerate G_0 -invariant bilinear form. Given the bilinear form, we can speak of nilpotent p -characters, i.e. those which correspond to nilpotent elements in \mathfrak{g}_0 under the isomorphism $\mathfrak{g}_0 \cong \mathfrak{g}_0^*$ induced by the bilinear form.

We are now ready to give an alternative proof of [WZ1, Theorem. 4.3].

Theorem 3.3. *Let \mathfrak{g} be as in Section 2.4, and let $\chi \in \mathfrak{g}_0^*$ be nilpotent. Write $d_0 | d_1 = \text{codim } \mathfrak{g}_\chi$. Then the dimension of every finite dimensional $U_\chi(\mathfrak{g})$ -module V is divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$.*

Proof. By [Jan2, Theorem 2.8.1], G_0 has finitely many orbits in \mathfrak{g}_0 . Thus G_0 has finitely many coadjoint orbits in \mathfrak{g}_0^* via the G_0 -equivariant isomorphism $\mathfrak{g}_0 \cong \mathfrak{g}_0^*$. If $\chi \in \mathfrak{g}_0^*$ is nilpotent, so is $K^\times \chi$. Then by [Jan2, Lemma 2.10], $K^\times \chi$ is contained in the G_0 -orbit of χ . Now since $\text{Ad}_{G_0} \subseteq G(\mathfrak{g}_0)$, we have $K^\times \chi \subseteq G_0 \cdot \chi \subseteq \Omega(\chi)$. Hence by Theorem 3.2 (ii), the dimension of every finite dimensional $U_\chi(\mathfrak{g})$ -module V is divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$, i.e., the super KW conjecture holds for $U_\chi(\mathfrak{g})$. \square

Remark 3.4. In a similar fashion as in Theorem 3.3, we can use Theorem 3.2 to give an alternative proof of super KW conjecture for the queer Lie superalgebra with nilpotent p -characters ([WZ2, Theorem 4.4]).

Now together with [WZ1, Remarks 2.5 and 4.6, Theorem 5.2], we have strengthened the super KW property for basic classical Lie superalgebras as follows. We remark here that [WZ1, Theorem 5.2] remains valid for basic classical Lie superalgebras with assumption on p as in Section 2.4.

Theorem 3.5 (Super Kac-Weisfeiler Conjecture). *Let \mathfrak{g} be a basic classical Lie superalgebra as in Section 2.4, and let $\chi \in \mathfrak{g}_0^*$. Let $d_0 | d_1 = \text{codim } \mathfrak{g}_\chi$. Then the dimension of every $U_\chi(\mathfrak{g})$ -module M is divisible by $p^{\frac{d_0}{2}} 2^{\lfloor \frac{d_1}{2} \rfloor}$.*

4. SEMISIMPLE p -CHARACTERS FOR BASIC CLASSICAL LIE SUPERALGEBRAS

Now we turn our attention to basic classical Lie superalgebras \mathfrak{g} (Sect. 2.4) with a semisimple p -character $\chi \in \mathfrak{g}_0^*$ (see below for a definition). Our purpose is to give a semisimplicity criterion for the reduced enveloping superalgebra $U_\chi(\mathfrak{g})$.

Let \mathfrak{g} be one of the basic classical Lie superalgebras as in Sect. 2.4. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} (and of \mathfrak{g}_0). It defines the set of roots $\Delta = \Delta_0 \cup \Delta_1$, where Δ_0 (resp. Δ_1) is the set of even (resp. odd) roots. Let W be the Weyl group of \mathfrak{g}_0 . The G_0 -invariant bilinear form on \mathfrak{g} induces a W -invariant bilinear form (\cdot, \cdot) on \mathfrak{h}^* . Put

$$\begin{aligned} \overline{\Delta}_0 &= \{\alpha \in \Delta_0 \mid \frac{1}{2}\alpha \notin \Delta_1\}; \\ \overline{\Delta}_1 &= \{\alpha \in \Delta_1 \mid 2\alpha \notin \Delta_0\} = \{\alpha \in \Delta_1 \mid (\alpha, \alpha) = 0\}. \end{aligned}$$

For $\alpha \in \Delta$, let H_α and X_α be a choice of coroot and root vectors respectively.

Let $\chi \in \mathfrak{g}_0^*$ be a p -character satisfying $\chi(X_\alpha) = 0$ for all $\alpha \in \Delta_{\bar{0}}$. A p -character which is $G_{\bar{0}}$ -conjugate to one of such χ is called *semisimple*.

4.1. Fix an arbitrary set of simple roots Π of Δ . It determines a set of positive roots ${}^\Pi\Delta^+$. Denote by ${}^\Pi\Delta_0^+$, ${}^\Pi\Delta_1^+$, ${}^\Pi\bar{\Delta}_0^+$, and ${}^\Pi\bar{\Delta}_1^+$ the subsets of positive roots in the sets $\Delta_{\bar{0}}$, $\Delta_{\bar{1}}$, etc. respectively. Let

$$\mathfrak{g} = {}^\Pi\mathfrak{n}^- \oplus \mathfrak{h} \oplus {}^\Pi\mathfrak{n}^+$$

be the corresponding triangular decomposition. Put ${}^\Pi\mathfrak{b} = \mathfrak{h} \oplus {}^\Pi\mathfrak{n}^+$. Let ${}^\Pi\rho = {}^\Pi\rho_{\bar{0}} - {}^\Pi\rho_{\bar{1}}$, where ${}^\Pi\rho_{\bar{0}}$ (resp. ${}^\Pi\rho_{\bar{1}}$) is the half sum of positive even (resp. odd) roots.

For $\lambda \in \Lambda_\chi := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\}$, the baby Verma module $Z_\chi^\Pi(\lambda)$ is defined to be

$$Z_\chi^\Pi(\lambda) := U_\chi(\mathfrak{g}) \otimes_{U_\chi({}^\Pi\mathfrak{b})} K_\lambda,$$

where K_λ is the one-dimensional $U_\chi({}^\Pi\mathfrak{b})$ -module upon which \mathfrak{h} acts via multiplication by λ and ${}^\Pi\mathfrak{n}^+$ acts as zero. Write $v_\lambda = 1 \otimes 1_\lambda$ in $Z_\chi^\Pi(\lambda)$.

Index roots in ${}^\Pi\Delta^+$ by $\{1, 2, \dots, N = |\Delta|/2\}$ in a way that is compatible with heights of roots. Here by compatible we mean that the shorter the root is in height the smaller it is indexed. For $\alpha \in {}^\Pi\Delta^+$, put

$$m_\alpha = \begin{cases} p-1 & \text{if } \alpha \in {}^\Pi\Delta_0^+; \\ 1 & \text{if } \alpha \in {}^\Pi\Delta_1^+. \end{cases}$$

Lemma 4.1. *Any nonzero submodule S of $Z_\chi^\Pi(\lambda)$ contains vector $X_{-\alpha_1}^{m_{\alpha_1}} \cdots X_{-\alpha_N}^{m_{\alpha_N}} v_\lambda$.*

Proof. The proof is similar to the proof of [Rud, Proposition 4] and will be skipped here. \square

Lemma 4.2. *In $U(\mathfrak{g})$, we have*

$$X_{\alpha_1}^{m_{\alpha_1}} \cdots X_{\alpha_N}^{m_{\alpha_N}} X_{-\alpha_1}^{m_{\alpha_1}} \cdots X_{-\alpha_N}^{m_{\alpha_N}} - {}^\Pi\Phi \in U(\mathfrak{g})^{{}^\Pi\mathfrak{n}^+},$$

where ${}^\Pi\Phi$ is a polynomial in $\{H_\alpha \mid \alpha \in \Pi\}$ of degree $(\frac{(p-1)|\Delta_{\bar{0}}|}{2} + \frac{|\Delta_{\bar{1}}|}{2})$.

Proof. The proof is similar to the proof of [Rud, Proposition 5] and will be omitted here. \square

Proposition 4.3. *The baby Verma module $Z_\chi^\Pi(\lambda)$ is irreducible if and only if ${}^\Pi\Phi(\lambda) \neq 0$ for $\lambda \in \Lambda_\chi$.*

Proof. Follows readily from Lemmas 4.1 and 4.2. \square

Finally in this subsection, put ${}^\Pi\Phi'(\lambda) = {}^\Pi\Phi(\lambda - {}^\Pi\rho)$ for $\lambda \in \mathfrak{h}^*$.

4.2. Retain the notations from previous subsection. Let $\delta \in \Pi$ be a simple root. Then it is one of the following three types:

- (i) $\delta \in \bar{\Delta}_0$;
- (ii) $\delta \in \bar{\Delta}_1$;
- (iii) $\delta \in \Delta_{\bar{1}} \setminus \bar{\Delta}_{\bar{1}}$ with $2\delta \in \Delta_{\bar{0}} \setminus \bar{\Delta}_{\bar{0}}$.

For such a δ , we shall denote

$$\delta^* = \begin{cases} \delta, & \text{in case (i) and (ii);} \\ \{\delta, 2\delta\}, & \text{in case (iii).} \end{cases}$$

Let r_δ be the (even or odd) reflection associated to δ . When δ is of type (i), r_δ is just the even reflection in \mathfrak{h}^* defined by

$$r_\delta(\lambda) = \lambda - \frac{2(\delta, \lambda)}{(\delta, \delta)}\delta, \quad \text{for } \lambda \in \mathfrak{h}^*. \quad (4.1)$$

When δ is of type (iii), r_δ is by definition the even reflection $r_{2\delta}$, which is also given by formula (4.1). When δ is of type (ii), r_δ is given by the following

$$r_\delta(\beta) = \begin{cases} -\delta, & \text{if } \beta = \delta; \\ \beta + \delta, & \text{if } (\delta, \beta) \neq 0; \\ \beta, & \text{if } \beta \neq \delta \text{ and } (\delta, \beta) = 0. \end{cases}$$

It is known that (see, for example, [Ser]) $r_\delta\Pi$ is the set of simple roots of the positive system ${}^{r_\delta\Pi}\Delta^+ := r_\delta({}^\Pi\Delta^+)$, $-\delta^* \in {}^{r_\delta\Pi}\Delta^+$, and ${}^{r_\delta\Pi}\Delta^+ \cap {}^\Pi\Delta^+ = {}^\Pi\Delta^+ \setminus \delta^*$.

By going through the same argument in the previous subsection, we know that there is a polynomial ${}^{r_\delta\Pi}\Phi$ on \mathfrak{h}^* of degree $(\frac{(p-1)|\Delta_{\tilde{0}}|}{2} + \frac{|\Delta_{\tilde{1}}|}{2})$ satisfying that ${}^{r_\delta\Pi}\Phi(\lambda) \neq 0$ if and only if the baby Verma module $Z_\chi^{r_\delta\Pi}(\lambda)$ associated to the positive system ${}^{r_\delta\Pi}\Delta^+$ is irreducible for any $\lambda \in \Lambda_\chi$.

For two polynomials f_1 and f_2 , write $f_1 \sim f_2$ if $f_1 = cf_2$ for some $c \in K^\times$.

Lemma 4.4. *We have ${}^{r_\delta\Pi}\Phi' \sim {}^\Pi\Phi'$ for a simple root $\delta \in \Pi$.*

Proof. Let us prove when δ is of type (iii), the other two cases can be proved in a similar fashion. First we observe that the vector $X_{-\delta}X_{-2\delta}^{p-1}v_\lambda$ in $Z_\chi^\Pi(\lambda)$ is annihilated by any root vector X_α for $\alpha \in {}^{r_\delta\Pi}\Delta^+$. It follows that there is a nontrivial $U(\mathfrak{g})$ -module homomorphism

$$Z_\chi^{r_\delta\Pi}(\lambda + \delta) \rightarrow Z_\chi^\Pi(\lambda).$$

Since the two baby Verma modules have the same dimension, $Z_\chi^{r_\delta\Pi}(\lambda + \delta)$ being reducible will imply that $Z_\chi^\Pi(\lambda)$ is reducible. By Proposition 4.3, we have ${}^{r_\delta\Pi}\Phi(\lambda + \delta)$ divides ${}^\Pi\Phi(\lambda)$, and so ${}^{r_\delta\Pi}\Phi(\lambda + \delta) \sim {}^\Pi\Phi(\lambda)$. Hence ${}^{r_\delta\Pi}\Phi'(\lambda) \sim {}^\Pi\Phi'(\lambda)$ since ${}^{r_\delta\Pi}\rho = {}^\Pi\rho - \delta$.

When δ is of type (i), then as in the classical case, the vector $X_{-\delta}^{p-1}v_\lambda$ in $Z_\chi^\Pi(\lambda)$ is a singular vector for the positive system ${}^{r_\delta\Pi}\Delta^+$. We then can argue the same way as for δ 's of type (iii).

When δ is of type (ii), we only need to observe that the vector $X_{-\delta}v_\lambda$ in $Z_\chi^\Pi(\lambda)$ is a singular vector for the positive system ${}^{r_\delta\Pi}\Delta^+$. The rest of the argument is the same as for δ 's of type (iii). □

Since by applying (even and odd) simple reflections, we are able to obtain any set $\tilde{\Pi}$ of simple roots from a given set Π of simple roots, we conclude by Lemma 4.4

that, the polynomial $\tilde{\Pi}\Phi'$ does not depend on the choice of $\tilde{\Pi}$ up to “ \sim ”-equivalence. Thus we can suppress the left superscript $\tilde{\Pi}$ of $\tilde{\Pi}\Phi'$ and write Φ' instead.

Proposition 4.5. *We have $\Phi'(\lambda) \sim \prod_{\alpha \in {}^{\Pi}\Delta_0^+} ((\lambda|\alpha)^{p-1} - 1) \cdot \prod_{\beta \in {}^{\Pi}\Delta_1^+} (\lambda|\beta)$, for any set of simple roots Π .*

A different choice of simple roots in Proposition 4.5 will only lead to a plus/minus sign in the product on the right hand side in the Proposition.

Proof. First observe that if $\Delta_{\bar{1}} \setminus \overline{\Delta_{\bar{1}}} \neq \emptyset$, then any $\delta \in \Delta_{\bar{1}} \setminus \overline{\Delta_{\bar{1}}}$ appears as a simple root in some set $\tilde{\Pi}$ of simple roots. The root vector X_δ generates an embedded $\mathfrak{osp}(1|2)$ in \mathfrak{g} . Consider the minimal parabolic subalgebra $\mathfrak{p} = \mathfrak{osp}(1|2) + {}^{\tilde{\Pi}}\mathfrak{b}$, and the induced module $Z_\chi^{\mathfrak{p}}(\lambda) = U_\chi(\mathfrak{p}) \otimes_{U_\chi({}^{\tilde{\Pi}}\mathfrak{b})} K_\lambda$. The $U_\chi(\mathfrak{p})$ -module $Z_\chi^{\mathfrak{p}}(\lambda)$ is merely the baby Verma module $Z_\chi^{\mathfrak{osp}(1|2)}(\lambda)$ of the embedded $\mathfrak{osp}(1|2)$ upon which \mathfrak{h} acts as weight multiplication by λ and X_α acts zero for $\alpha \in {}^{\tilde{\Pi}}\Delta^+ \setminus \{\delta, 2\delta\}$. By the transitivity of induced modules, we have

$$Z_\chi^{\tilde{\Pi}}(\lambda) \cong U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} Z_\chi^{\mathfrak{p}}(\lambda).$$

It follows from [WZ1, Section 6.5] that if λ satisfies $(\lambda + {}^{\tilde{\Pi}}\rho|\delta)^p - (\lambda + {}^{\tilde{\Pi}}\rho|\delta) = 0$, then $Z_\chi^{\mathfrak{osp}(1|2)}(\lambda)$ is reducible; hence $Z_\chi^{\mathfrak{p}}(\lambda)$ and so $Z_\chi^{\tilde{\Pi}}(\lambda)$ will be reducible. By Proposition 4.3, we have $(\lambda + {}^{\tilde{\Pi}}\rho|\delta)^p - (\lambda + {}^{\tilde{\Pi}}\rho|\delta)$ divides $\tilde{\Pi}\Phi(\lambda)$, that is, $((\lambda|\delta)^p - (\lambda|\delta))$ divides Φ' . Note that for two such roots δ and δ' , $((\lambda|\delta)^p - (\lambda|\delta))$ and $((\lambda|\delta')^p - (\lambda|\delta'))$ are coprime if $\delta \neq \pm\delta'$. Since for any such root δ either δ or $-\delta$ is in ${}^{\Pi}\Delta_1^+$, and since the arbitrary choice of such δ we conclude that

$$\prod_{\delta \in {}^{\Pi}\Delta_1^+ \setminus {}^{\Pi}\overline{\Delta_1^+}} (\lambda|\delta) \cdot \prod_{2\delta \in {}^{\Pi}\Delta_0^+ \setminus {}^{\Pi}\overline{\Delta_0^+}} ((\lambda|2\delta)^{p-1} - 1) \text{ divides } \Phi'.$$

Next observe that any odd root $\beta \in \overline{\Delta_{\bar{1}}}$ (of type (ii)) appears in some set of simple roots. The root vector X_β generates an embedded $\mathfrak{sl}(1|1)$. Using similar arguments as for type (iii) simple roots above, we can show that

$$\prod_{\beta \in {}^{\Pi}\overline{\Delta_1^+}} (\lambda|\beta) \text{ divides } \Phi'.$$

In the proof, we need an irreducibility criterion for $\mathfrak{sl}(1|1)$ -baby Verma modules, which can be easily deduced from that for $\mathfrak{gl}(1|1)$ -baby Verma modules as in [WZ2, Proposition 7.7].

For roots in $\overline{\Delta_0}$ (of type (i)), in a similar but classical manner (cf. [Rud, Proof of Proposition 6]), we can show that

$$\prod_{\alpha \in {}^{\Pi}\overline{\Delta_0^+}} ((\lambda|\alpha)^{p-1} - 1) \text{ divides } \Phi'.$$

Finally, the Proposition follows from a degree consideration and the fact that the above three factors are mutually coprime. \square

Theorem 4.6. *A baby Verma module $Z_\chi^\Pi(\lambda)$ for $\lambda \in \Lambda_\chi$ is irreducible if and only if*

$$\prod_{\alpha \in {}^\Pi\Delta_0^+} ((\lambda + {}^\Pi\rho | \alpha)^{p-1} - 1) \cdot \prod_{\beta \in {}^\Pi\Delta_1^+} (\lambda + {}^\Pi\rho | \beta) \neq 0.$$

Proof. Follows readily from Propositions 4.3 and 4.5. \square

Theorem 4.7. *The algebra $U_\chi(\mathfrak{g})$ is a semisimple algebra if and only if χ is regular semisimple.*

Proof. The argument, which uses the irreducibility criterion in Theorem 4.6, is pretty standard. We include it here just for the sake of completeness.

Since χ satisfies $\chi(X_\alpha) = 0$ for each $\alpha \in \Delta_0$, for any set of simple roots Π , the baby Verma modules $Z_\chi^\Pi(\lambda)$ for $\lambda \in \Lambda_\chi$ have unique irreducible quotients, and they form a complete and irredundant set of irreducible $U_\chi(\mathfrak{g})$ -modules. Now by Wedderburn Theorem and a dimension counting argument, $U_\chi(\mathfrak{g})$ is semisimple if and only if all the baby Verma modules $Z_\chi^\Pi(\lambda)$ for $\lambda \in \Lambda_\chi$ are simple. By Theorem 4.6, $Z_\chi^\Pi(\lambda)$ being simple for all $\lambda \in \Lambda_\chi$ is equivalent to ${}^\Pi\Phi(\lambda) \neq 0$ for all $\lambda \in \Lambda_\chi$, which in turn is equivalent to (i) $(\lambda + {}^\Pi\rho)(H_\alpha) \notin \mathbb{F}_p \setminus \{0\}$ for all $\alpha \in \Delta_0$ and (ii) $(\lambda + {}^\Pi\rho)(H_\beta) \neq 0$ for all $\beta \in \Delta_1$.

Recall that under current assumption, χ is regular semisimple if and only if $\chi(H_\alpha) \neq 0$ for all $\alpha \in \Delta$. If χ is regular semisimple, then it follows that for any $\lambda \in \Lambda_\chi$, $\lambda(H_\alpha) \notin \mathbb{F}_p$ for all $\alpha \in \Delta$ since $\lambda(H_\alpha)^p - \lambda(H_\alpha) = \chi(H_\alpha)^p$. In this situation, both (i) and (ii) are true since ${}^\Pi\rho(H_\alpha) \in \mathbb{F}_p$ for any $\alpha \in \Delta$. Hence all $Z_\chi^\Pi(\lambda)$ are simple and $U_\chi(\mathfrak{g})$ is semisimple.

Conversely, if χ is not regular semisimple, then $\chi(H_\alpha) = 0$ for some $\alpha \in \Delta$. Let us assume $\alpha \in \Delta_0$, since the other case can be argued in a similar fashion. Then $\lambda(H_\alpha) \in \mathbb{F}_p$ for $\lambda \in \Lambda_\chi$. Since shifting the value of $\lambda(H_\alpha)$ by a number in \mathbb{F}_p will still result in an element in Λ_χ (noting that the values of $\lambda(H_\beta)$ for some $\beta \in \Delta$ will be changing correspondingly), we may thus assume $(\lambda + {}^\Pi\rho)(H_\alpha) = 1$. Then ${}^\Pi\Phi(\lambda) = 0$ and $Z_\chi^\Pi(\lambda)$ is reducible by Theorem 4.6. Hence $U_\chi(\mathfrak{g})$ is not semisimple. \square

Remark 4.8. Note that the “if” part of the theorem is a consequence (cf. [WZ1, Corollary 5.7]) of the Super Kac-Weisfeiler Conjecture (Theorem 3.5).

Also, for type I basic classical Lie superalgebras, Theorems 4.6 and 4.7 are consequences of an equivalence of categories between typical $U_\chi(\mathfrak{g})$ -modules and typical $U_\chi(\mathfrak{g}_0)$ -modules (see [Z, Theorems 4.1 and 4.3]).

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